Examination of stability of the computer tomography algorithms in case of the incomplete information for the objects with non-transparent elements

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Abstract. The presented paper is a continuation of research devoted to the investigation of efficiency of some reconstructive computer tomography algorithms with the incomplete input data set. The new issues discussed in this elaboration are the results concerning the stability of these algorithms applied for examining the objects with non-transparent elements. In paper [14] the effectiveness of the considered algorithms has been also investigated, however the problem of stability was omitted. In the current paper we return to this problem.

Keywords: computer tomography, reconstructive algorithms, parallel algorithms, chaotic algorithms, numerical techniques.

2010 Mathematics Subject Classification: 11Y50, 11Y55, 11Y60.

1. Introduction

Aim of the computer tomography is the reconstruction of the internal structure of an object without disturbing its construction, on the basis of examination with the aid of hard radiation rays (projection). The computer tomography is recently applied in many fields. The most known area of its application is certainly the medicine – the present-day computer tomographs, used in medicine, are able to execute over million of projections and by the resolution at the level of micrometers they may obtain the complete reconstruction within the time less than one minute. Such good results can be received not only because of the efficient algorithms but also thanks to the proper specification of the problem enabling to apply the most useful algorithm. However, there are some situations when there is no possibility to execute the required number of projections or their quality is not good enough or even both. The good example

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R. Wituła, B. Bajorska-Harapińska, E. Hetmaniok, D. Słota, T. Trawiński (eds.), *Selected Problems on Experimental Mathematics*. Wydawnictwo Politechniki Śląskiej, Gliwice 2017, pp. 39–59.

of such situation is the examination of the coal layer before its exploitation. Aim of this examination is to uncover some potential dangers which may threaten the health and safety of the workers (for example, the natural reservoirs of compressed gas) or to detect some economically undesirable geological inclusions (for example, the high content of rocks in the coal layers). With respect to the limited access to the coal layer as well as to its size, the application of classical algorithms, used in the medical tomography, may be impossible. Therefore one should either introduce some adaptations of these algorithms by taking into account the specificity of such problems or else, some new algorithms should be proposed. In the second case the usefulness of such algorithms suppose to be certainly investigated.

2. Ideas of the computer tomography

Possibility of using the computer tomography algorithms is grounded, among others, on the fact that each kind of material is characterized by some capability for absorbing the energy of the hard radiation ray. When we expose the given object to the hard radiation, the ray of this radiation loses its initial energy proportionally to the sum of covered distances within this object and proportionally to the, mentioned before, absorbtion coefficient of the material, the object is made of. The equation expressing these losses (for the two-dimensional tomography) on the strait line road L may be of the form

$$p \equiv \ln \frac{I_0}{I} = \int_L f(x,y)dL, \tag{1}$$

where $f(x,y)$ is the function of density distribution (function describing the internal structure of the object), I_0 denotes the initial intensity of radiation, I means the final intensity and p denotes the projection.

In the early 20th century the Austrian mathematician Johann Radon proved in paper [33] that the image (two-dimensional as well as three-dimensional) of a given object can be reconstructed on the basis of infinite number of projections and the formula for determining this reconstruction can be found in the mentioned Radon's paper [33], as well as in the paper written by Sigurdur Helgason [23]. Obviously in reality we are not able to get the infinite number of projections, therefore we have to base on the data obtained from the finite number of projections (determined on the ground of finite number of scanning angles and the finite number of rays in one beam). Frank Natterer in paper [30] showed that the formulas derived by Radon and Helgason do not guarantee the unique reconstruction of the structure of a considered object. In the mentioned paper, and also in papers [27, 28], there are given the conditions connecting the reconstructed function with the number of projections required to ensure the uniqueness of reconstruction.

Certainly the determination of the projection, itself, do not pose the problems (especially from the mathematical point of view) – in practise one just exposes the given object to the radiation and then one measures, according to the idea taken from formula (1), the ratio of the final and initial intensity of radiation. Much more difficult task consists in inverting this process, that is in reconstructing the function $f(x,y)$ on the basis of obtained projections p. This kind of task is solved with the aid of appropriate algorithms which can be divided into two groups: the analytic algorithms and the algebraic algorithms. The first group, including the analytic algorithms, is grounded on such concepts as, among others, the Fourier transform, the Radon transformation and the reverse projection (see, for example, [4, 17, 26]). Unfortunately, the assumptions which must be satisfied for the analytic algorithms, especially for the incomplete input data, do not make possible to use these algorithms in case of problems investigated in this paper (for details please see the paper [32]).

Thus, for solving the problems with the incomplete information, discussed in section 3, the algebraic algorithms must be applied. Of course, the possibility of using these algorithms must be first verified, but it is already done, for example in [21, 22].

3. Problem of the incomplete information

In the classical algorithms of computer tomography it is assumed that the projections can be obtained at the sufficiently enough number of scanning angles (for the beams of rays formed in the shape of fan or for the parallel beams, because in case of the modern spiral medical computer tomographs the concept of the scanning angle is not so clear any more, and for the sufficiently enough number of rays in one beam. In some technical problems, because of the size, localization or limited access to the investigated object, the assumption about the number of projections and their "quality" (the scanning angles) cannot be fulfilled. An example of such situation, as we mentioned already in the Introduction, is the examination of the coal layer when the access to this layer is strongly limited. In such case we will distinguish two systems of collecting the projections. The first considered system is the $1 \times 1, 1 \times 1$ system, in which the sources and the detectors are located on the opposite sides of the examined object but the radiation beam is detected only on the side opposite to the side where the transmitters are placed. The second system, especially important in examination of such objects like the coal layer, the access to which is only from two of its sides, is the 1×1 system. In this system the ray transmitters are situated only on one side of the investigated object and the detectors are located only on the opposite side. Both systems are illustrated in Figure 1, in which symbol 1 denotes the sources (transmitters), symbol 2 – the examined object, 3 – the rays and 4 denotes the detectors.

As we can see, especially in case of system 1×1, the data received in the described way deviate significantly from the required information – the data are strongly "one-sided" and the number of data is strongly limited (with regard to the specific conditions in the coal mines). Therefore we call such case as the problem of the incomplete information (incomplete set of input data).

As we mentioned before, the existing algorithms may fail in the above described situation. It has been proven (experimentally) that the analytic algorithms do not deal with the problems of incomplete information, whereas the algebraic algorithms, properly optimized and adapted, may be applied, theoretically at least, for solving these kinds of problems, also in case of objects with non-transparent elements. In order to justify the practical usage of these algorithms their stability should be investigated, that is their resistance to the noises, and it is the purpose of this elaboration.

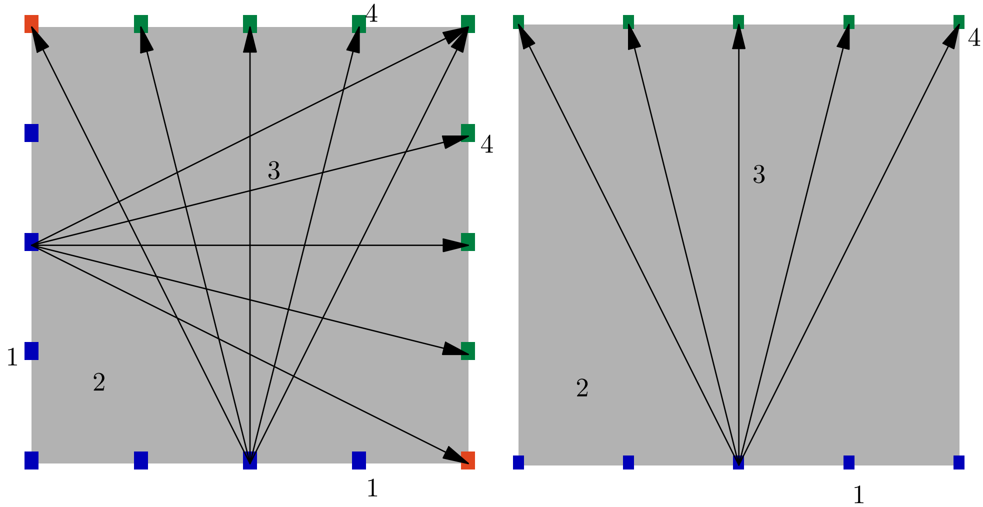

Fig. 1. Systems of collecting the data: system $1 \times 1, 1 \times 1$ and system 1×1

4. Algebraic algorithms

In class of the algebraic algorithms *ART* (Algebraic Reconstruction Techniques – to learn more about these algorithms please see, for example, the papers [1, 4, 13, 15, 17, 32]) we assume, for the start, that we introduce the discretization of the region containing the examined object (the most often having the form of a square) into the sufficiently big number of congruent squares (pixels) so that the function $f(x, y)$, describing the distribution of density, in each of these discretization squares possesses some constant (unknown) value and so that the reconstructed function $f(x, y)$ can be presented in the form of the following finite linear combination of the distribution constants and the basis functions

$$f(x, y) \approx \sum_{i=1}^{N} f_i b_i(x, y), \tag{2}$$

where $f_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, are the unknown (constant) coefficients of the distribution, $\{b_i(x, y)\}$, $i = 1, 2, \ldots, N$, are the basis functions and $N = n^2$ denotes the number of pixels (n describes the density of discretization). In our case the set of basis functions is of the form

$$b_i(x, y) = \begin{cases} 1, & \text{when point } (x, y) \text{ belongs to the interior of the } i\text{-th pixel}, \\ 0, & \text{in opposite case.} \end{cases}$$

By applying the Radon transform to equation (2) (in the discrete case, when we have in formula (1) the straight lines L_j representing the single rays) we get

$$p_j \approx \sum_{i=1}^{N} f_i a_{ij}, \tag{3}$$

where p_j, $j = 1, 2, \ldots, m$, denote the projection values, m means the number of projections, f_i, $i = 1, 2, \ldots, N$, are the distribution coefficients, $N = n^2$ denotes the number of pixels and a_{ij} describes (geometrically) the length of the common part of the j-th ray and the i-th pixel. Certainly, there is many sets of functions which can be selected as the set of basis functions (see, for example, [24, 31]) depending on the physical character of the investigated phenomenon.

The obtained formula (3) gives the ground for the family of algebraic algorithms. On its basis the following system of equations is constructed

$$\mathbf{A}\mathbf{x} = \mathbf{p}, \tag{4}$$

where $\mathbf{A} = (a_{ij})$ is the coefficient matrix of dimension $m \times N$ (each element of this matrix denotes the length of the common part of the j-th ray and the i-th pixel), $\mathbf{x} = \{x_1, x_2, \ldots, x_N\}^T$ is the vector of unknown elements (each element of this vector denotes the value of the density distribution function in the given pixel), $\mathbf{p} = \{p_1, p_2, \ldots, p_m\}^T$ means the projection vector (each element of this vector denotes the value of the successive projection connected with the total loss of energy of the given ray) and T denotes the transpose operation.

As we have mentioned before, the construction of the above system of equations is the common part of the family of algebraic algorithms, whereas the difference between these algorithms lies in the method of solving this system.

4.1. Algorithms ART and ART-3

The coefficient matrix $\mathbf{A}$ of the system of equations (4) has certain features causing that the application of classical methods for solving the systems of linear equations became definitely ineffective or even impossible. These features are, among others, the dimension – the considered matrix is strongly non-square matrix of a very big dimension, it has much more rows than columns, the matrix is the sparse matrix, that is the most of its elements are the zero elements and the non-zero elements are arranged in non-ordered way. Algorithms discussed in this section are based on the method of solving the systems of linear equations proposed in 1937 by Stefan Kaczmarz [25], which certainly was not originally connected with the computer tomography. The author proved the convergence of developed algorithms for the systems with a (unique) solution under additional assumption that the matrix of the solved system of equations is the square matrix. The main idea of the Kaczmarz algorithm is illustrated in Figure 2.

So, the considered algorithm consists in sequential projecting the previous approximation of the solution on the successive hyperplanes under the assumption that the initial (zero) solution $\mathbf{x}^{(0)}$ is fixed and as the hyperplane H_i we understand here each row of system (4):

$$H_i = \{\mathbf{x} \in \mathbb{R}^N : (\mathbf{a}^i, \mathbf{x}) = p_i\},$$

where $\mathbf{a}^i$, $i = 1, 2, \ldots, m$, is the i-th row of matrix $\mathbf{A}$, p_i, $i = 1, 2, \ldots, m$, denotes the i-th projection and symbol $(\cdot, \cdot)$ means the classic scalar product of vectors from space $\mathbb{R}^N$.

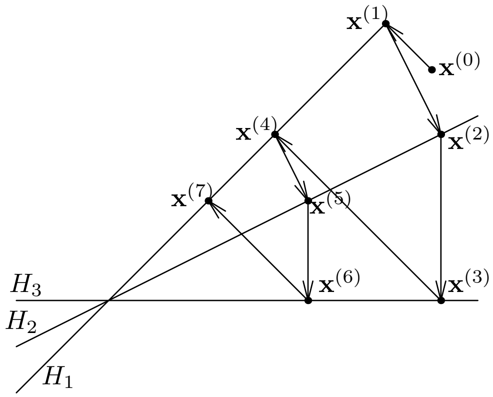

Fig. 2. Geometric interpretation of the Kaczmarz algorithm

Algorithm *ART* is the special case of the algorithm developed 33 years later by Gordon, Bender and Herman [13] for the needs of computer tomography, based on the Kaczmarz algorithm and applied for the three-dimensional problem. Tanabe in [34] proved the convergence of this algorithm also for the rectangular systems which gave the mathematical grounds for applying this procedure for the problems of computer tomography.

Algorithm *ART* can be presented in the following steps:
— selection of the initial solution which can be realized in many ways, for instance by taking into account some information about the expected solution (for example, we may know that no element of the solution vector can be negative);
— determination of the successive solutions by using the formula

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \lambda_k \frac{p_i - (\mathbf{a}^i, \mathbf{x}^{(k)})}{\|\mathbf{a}^i\|^2} \mathbf{a}^i, \tag{5}$$

where $\mathbf{x}^{(k)}$, $k = 1, 2, \ldots$, denotes the successive solution (more precisely – the successive approximation of solution, however we use here, for shortness, the term “solution”), $\mathbf{a}^i$, $i = 1, 2, \ldots, m$, is the i-th row of matrix $\mathbf{A}$, p_i, $i = 1, 2, \ldots, m$, denotes the i-th projection, symbol $(\cdot, \cdot)$ means the classic scalar product of vectors from space $\mathbb{R}^N$, symbol $\| \cdot \|$ describes the norm of vector form space $\mathbb{R}^N$ (in our case it is the length of vector), λ_k denotes the relaxation coefficient and $i = k(\mathrm{mod}\ m) + 1$;
— verification for the stop condition which may be, for example, executing the assumed number of projections or checking whether the new solution differs significantly from the previous one.

The described algorithm differs from the Kaczmarz algorithm in the application of the relaxation coefficient λ_k. By taking $\lambda_k = \lambda = 1$ we have exactly the Kaczmarz algorithm. Mathematically, for the fixed value of λ we deal with the respective homothety, including its special case – the orthogonal projection – for $\lambda = 1$. In the mentioned above papers it is proven that if $\lambda_k = \lambda$ for each k, then the ART algorithm is convergent for $0 < \lambda < 2$, whereas Trummer in paper [35] showed the convergence of this algorithm for the varying relaxation coefficient λ_k by proving that the condition for convergence is to satisfy the inequalities $0 < \liminf \lambda_k \leq \limsup \lambda_k < 2$ (obviously, the convergence means here the convergence of the given system of linear equations (4) to its solution).

Algorithm *ART*-3 differs from algorithm *ART* in the assumption of possibility for obtaining the perturbed projections (information about the sources of such perturbations and the methods of dealing with them can be found, among others, in [12, 29]). So, instead of the system of equations (4) we solve the analogical system of inequalities

$$\mathbf{p} - \mathbf{e} \leq \mathbf{A}\mathbf{x} \leq \mathbf{p} + \mathbf{e},$$

where $\mathbf{e} = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m]$ denotes the vector of perturbations (noises) of the projection.

Algorithm *ART*-3 runs analogically like algorithm *ART* in three steps, however the second step is different because the "projection" is realized according to the following formula

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + S_k \frac{\mathbf{a}^i}{\|\mathbf{a}^i\|^2},$$

where

$$S_k = \begin{cases} 0, & \text{for } |p_i - (\mathbf{a}^i, \mathbf{x}^{(k)})| \leq \varepsilon_i; \\ p_i - (\mathbf{a}^i, \mathbf{x}^{(k)}), & \text{for } |p_i - (\mathbf{a}^i, \mathbf{x}^{(k)})| \geq \varepsilon_i; \\ 2(p_i + \varepsilon_i - (\mathbf{a}^i, \mathbf{x}^{(k)})), & \text{for } p_i + \varepsilon_i < (\mathbf{a}^i, \mathbf{x}^{(k)}) < p_i + 2\varepsilon_i; \\ 2(-p_i + \varepsilon_i + (\mathbf{a}^i, \mathbf{x}^{(k)})), & \text{for } p_i - 2\varepsilon_i < (\mathbf{a}^i, \mathbf{x}^{(k)}) < p_i - \varepsilon_i, \end{cases}$$

where $\mathbf{x}^{(k)}$, $k = 1, 2, \ldots$, denotes the successive solution, $\mathbf{a}^i$, $i = 1, 2, \ldots, m$, is the i-th row of matrix $\mathbf{A}$, p_i, $i = 1, 2, \ldots, m$, denotes the i-th projection, symbol $(\cdot, \cdot)$ means the classic scalar product of vectors from space $\mathbb{R}^N$, symbol $\|\cdot\|$ describes the norm of vector from space $\mathbb{R}^N$, ε_i, $i = 1, 2, \ldots, m$, denotes the i-th element of the noise vector and $i = k(\text{mod } m) + 1$.

4.2. Other algebraic algorithms

Apart from the algorithms listed in the previous section, the group of the most popular algebraic algorithms includes also the *MART* algorithm (Multiplicative Algebraic Reconstruction Technique – more precisely, this is the whole family of algorithms) which has been examined with respect to its usefulness (the condition for convergence is the inequality $0 < \lambda_k^i a_{ij} \leq 1$), among others, in [6, 8]. This algorithm differs from the algorithms belonging to the *ART* group in such a way that the successive approximation is created not in the form of a full vector, but as its successive coordinates (additionally, the *ART* algorithms are additive, whereas the *MART* algorithm is multiplicative). The *MART* algorithm can be expressed in three steps, in such a way that the first and third steps are the same as previously presented and the second step is expressed by formula

$$x_j^{(k+1)} = \left(\frac{p_i}{(\mathbf{a}^i, \mathbf{x}^{(k)})}\right)^{\lambda_k^i a_{ij}} x_j^{(k)},$$

where $\mathbf{x}^{(k)}$, $k = 1, 2, \ldots$, denotes the next solution, $x_j^{(k+1)}$, $k = 1, 2, \ldots$, $j = 1, 2, \ldots, N$, is the j-th coordinate of the next solution, $\mathbf{a}^i$, $i = 1, 2, \ldots, m$, denotes the i-th row of matrix $\mathbf{A}$, p_i, $i = 1, 2, \ldots, m$, means the i-th projection, symbol $(\cdot, \cdot)$ denotes the classic scalar product of vectors from space $\mathbb{R}^N$, a_{ij} is the respective element of the coefficient matrix, λ_k^i denotes the relaxation coefficient and finally $i = k(\text{mod } m) + 1$.

Among the other well known algebraic algorithms we may also notice, for example, the simultaneous algorithms *SITR* (Simultaneous Iterative Reconstruction Techniques) which use all the rows of matrix A for creating the next solution (see [11, 31]), their improvement – the *SART* algorithm introduced in [2] or the group of summation algorithms including, among others, the *SUM* and *MSUM* algorithms.

The next important group is the group of algorithms basing on the described above algorithms but modifying them in order to improve their working (the most often by increasing their speed of convergence). The iterative reconstruction algorithms can be divided into the following groups (see [6]):

— the serial algorithms (discussed above algorithms *ART*, *ART*-3 and *MART*) – in these algorithms the order of hyperplanes, on the basis of which the successive solutions are computed, is predetermined;

— the parallel algorithms basing on the serial algorithms but increasing their working speed thanks to, for example, the parallel computations – the examples of such algorithms *SIRT* or *SMART* (see [36]). Algorithms of this kind require a quite big number of processors (equal to the number of equations in the solved system of equations (4)). If this requirement is difficult to fulfill, then we may apply the algorithms belonging to the next group;

— the block algorithms which can be divided into two groups: the serial block algorithms and the parallel block algorithms (see [5, 17, 32]);

— the chaotic algorithms (representing the special case of the asynchronous algorithms, mathematical grounds of which are studied, among others, in papers [3, 7, 10, 34, 16]), basing on the first listed here group of the iterative algorithms, however the choice of the order of the successive hyperplanes can be variable (or even random);

— the chaotic-block algorithms created in result of connecting both above described ideas.

The main difference between the serial algorithms and the parallel block algorithms results from the number of processors or the number of parallel threads which may be used in reconstructing the given object. So, when we have the possibility of using quite big number of processors or parallel threads (for example, in case of executing the computations basing on the processors of the graphics card), we split the matrix $\mathbf{A}$ of system (4) into the blocks so that the number of rows of the coefficient matrix is not bigger than the number of available possible threads. Within the given block the appropriate algorithm is executed simultaneously for each row, the obtained solutions are properly averaged creating the initial solution for the next block. Whereas, when we have the possibility of executing much lower amount of parallel computations (for example, with the aid of the threads of the central processor), we split the coefficient matrix into blocks, the number of which corresponds with the number of available parallel processors. Within each block the classic algebraic algorithm is sequentially executed with the assumption of simultaneous parallel work of all blocks. The obtained solutions (coming from each block) are properly averaged creating the initial solution for all blocks in the next iteration. Both described approaches are illustrated on the diagrams presented in the below figures (the serial block algorithm in Figure 3 and the parallel block algorithm in Figure 4.

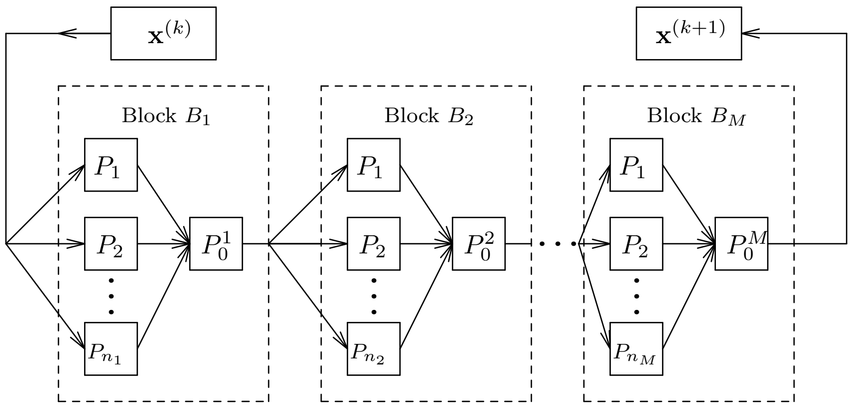

Fig. 3. General scheme of the serial block algorithm

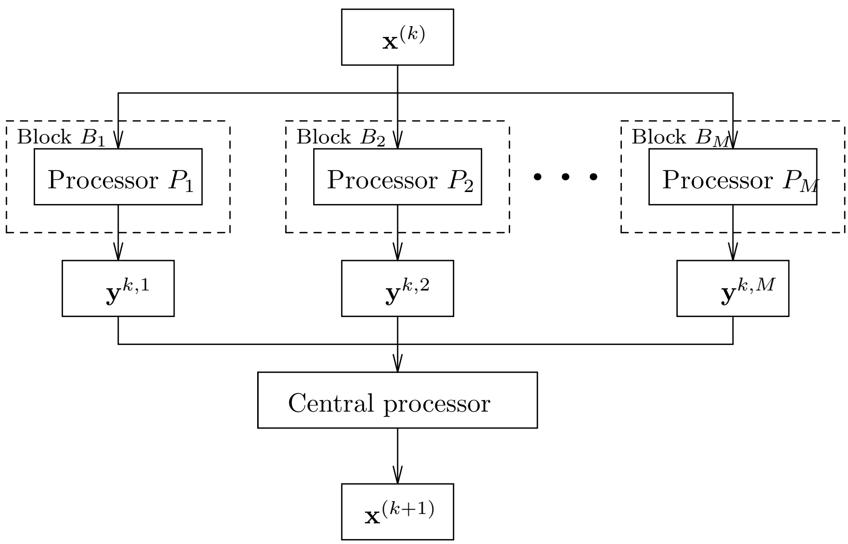

Fig. 4. General scheme of the parallel block algorithm

5. Previous results

The first question, that should be answered while investigating the usefulness of the even simple basic algebraic algorithms, is whether they can be applied for the discussed here problem of the incomplete input data 3. It appears that, after some proper adaptation of these algorithms, it is possible to select their parameters to obtain, quickly enough, the solution of satisfying quality, even in case of the three-dimensional problem (see [21, 22, 32]). In Figures 5 and 6 there are presented some exemplary results of these algorithms execution. These algorithms, as well as the other discussed here algorithms, are tested on some class of discrete functions strictly connected with the specificity of examined object, which may be the coal layer. With respect to this, the two-dimensional function of the density distribution, used for simulating the problem of considered kind, is expressed in the following form

$$f(x,y) = \begin{cases} c_1, & (x,y) \in D_1 \subset E \subset \mathbb{R}^2, \\ c_2, & (x,y) \in D_2 \subset E \subset \mathbb{R}^2, \\ \vdots & \vdots \\ c_n, & (x,y) \in D_n \subset E \subset \mathbb{R}^2, \\ 0, & \text{elsewhere}, \end{cases}$$

where regions D_i, $i = 1, 2, \ldots, n$, are mutually disjoint and region E, within which the examined object is located, is here in the form of a square: $E = \{(x,y)\colon -1 \leq x, y \leq 1\}$.

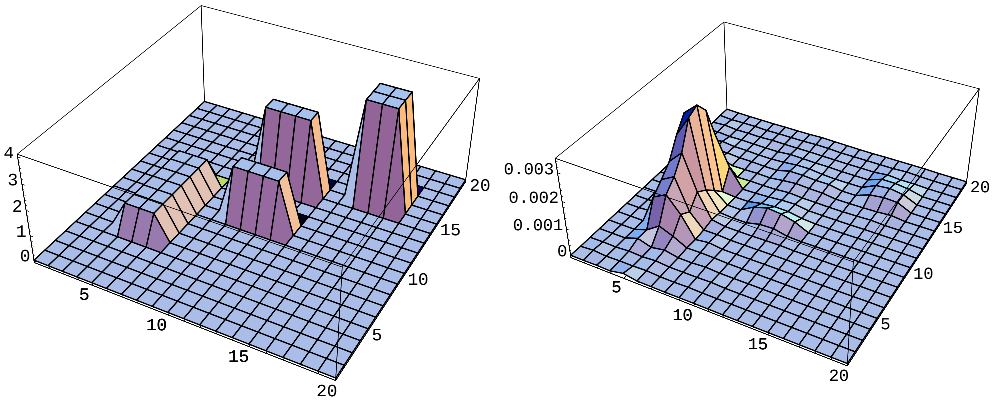

Fig. 5. Three-dimensional plots of reconstructions, together with the errors, obtained in simulation of algorithm ART-3 in system (1×1) for parameters $n = 20$, $pkt = 28$, $iter = 250$

In Figures 5 and 6 element n denotes the density of discretization (implying the number of pixels $N = n^2$), pkt means the number of sources and detectors (implying the value $m = pkt^2$) and $iter$ describes the number of iterations (one iteration means here the application of the algorithm for all the rows of matrix A). The right-hand-side figures present the plots of the absolute errors of the obtained reconstructions calculated as the absolute values of difference between the exact value and the reconstructed value for each of N pixels.

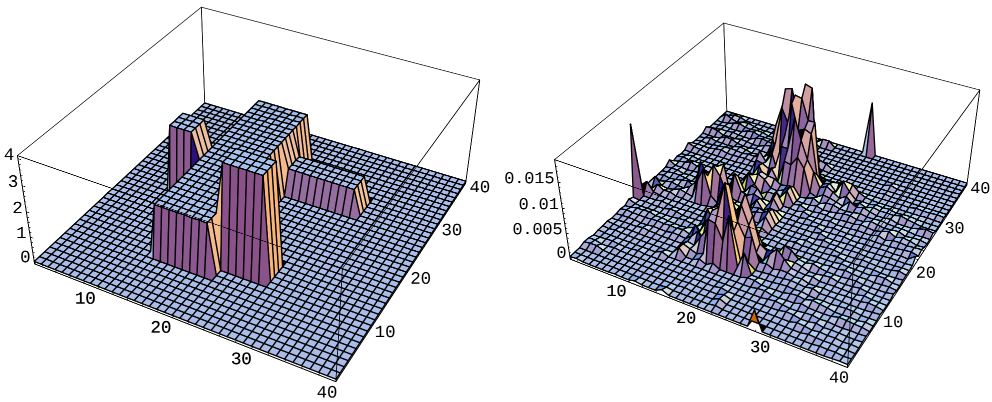

Fig. 6. Three-dimensional plots of reconstructions, together with the errors, obtained in simulation of algorithm ART-3 in system $(1 \times 1, 1 \times 1)$ for parameters $n = 40$, $pkt = 28$, $iter = 35$

The next examination concerned the possibility of practical usage of the, described in previous section, chaotic, block (parallel) and chaotic-block algorithms (like, for example, the algorithms *SZB*-3, *RB*-3, *CHART*-3, *CHRB*-3 – some of these algorithms have been proposed by the authors) and then, after positive verification, their comparison with classical algorithms. The comparative analysis and the stability examination have been executed and the results can be found, among others, in papers [18, 19, 20, 32]. In paper [14] the possibility of applying the mentioned algorithms for investigating the objects with non-transparent elements is also tested, however the examination of stability of the considered algorithms in solving problems of this kind has not been done till now.

6. Stability examination

Considering the real physical problems we are obviously not able to gain the 100-percent exact measurements. It is caused the most often by the inaccuracy of measuring devices or by the simplified mathematical model of the physical phenomenon. For example, in the computer tomography we assume that the trajectory of the ray movement has the form of a segment. In real, because of the non-homogeneity of the investigated object, the rays can be subject to dispersions or reflections.

The deviations, with respect to the values resulting from the theoretical considerations, cased by the mentioned above elements will be called the noises. Two the best known kinds of noises are the white noise (of the uniform distribution) and the normal noise (of the normal distribution). Considering the physical problems we do not usually deal with the white noise, therefore we focus now on the normal noise.

To simulate now this problem we will perturb the projection vector by the normal noise generated on the basis of normal distribution with the fixed parameter $\mu = 0$ and with the various values of parameter σ connected with range of noise. The generated vector of noises we will be added then to the projection vector and in this way we obtain the properly perturbed simulated measurements.

Among others in papers [14, 18, 19, 20, 32] one can find the examination of effectiveness of some classical and non-classical algorithms of computer tomography. The performed research revealed that the investigated algorithms are stable in such a sense that the reconstruction errors do not exceed the level of the range of noise. In the below included figures there are presented the selected results illustrating the stability of discussed algorithms.

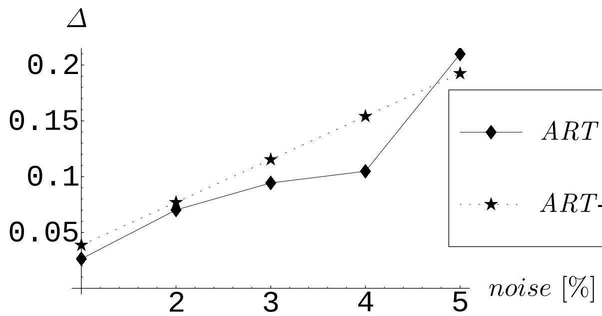

Fig. 7. Comparison of the impact of the noise on error Δ in simulation of algorithms ART-3 and ART in system $(1 \times 1, 1 \times 1)$

As we can see, the error Δ determined from the formula $\Delta = \max_{1 \le i \le N} \left| f_i - \tilde{f}_i \right|$, where N is the number of pixels, f_i denotes the exact value of function describing the density distribution in the i-th pixel and $\tilde{f}_i$ means the reconstructed value of function describing the density distribution in the i-th pixel, increases linearly together with the increasing range of normal noise. Similar situation takes place in case of algorithm $MART$, as well as for the discussed algorithms simulated in system (1×1).

Analyzing the above figures we may observe that the investigated block, chaotic and chaotic-block algorithms are stable as well. In the presented figures s denotes the level of normal noise, δ describes the maximal percentage error, that is $\delta = \frac{\Delta}{\max_{1 \le i \le N} |f_i|}$, n is the discretization density ($N = n^2$ means the number of pixels) and pkt denotes the number of sources (detectors) on one side (which implies the number of projections and, in consequence, in dependance on the system of collecting the data, the number of rows in system of equations (4)).

Let us proceed now to investigation of stability of the computer tomography algorithms applied for examination of objects with the non-transparent elements. Similarly as before we intend to select various functions describing the density distribution, various systems of collecting the data, various algorithms and various locations of the non-transparent element. So, for the system $(1 \times 1, 1 \times 1)$, for $n = 20$, $pkt = 18$ and 25 iterations, by using algorithm ART and by perturbing the projections with the normal noise, with the non-transparent element located at point $(-0.1, 0.2)$, we received the results presented in Figures 9–11.

Presented results show that even for the 5% noise the received reconstruction is of satisfying quality. In Figure 12 there is displayed the relation between the error Δ

and the number of iterations received for the various ranges of the noise (the ranges of noise, as well as the other parameters, are the same as in the previously presented results).

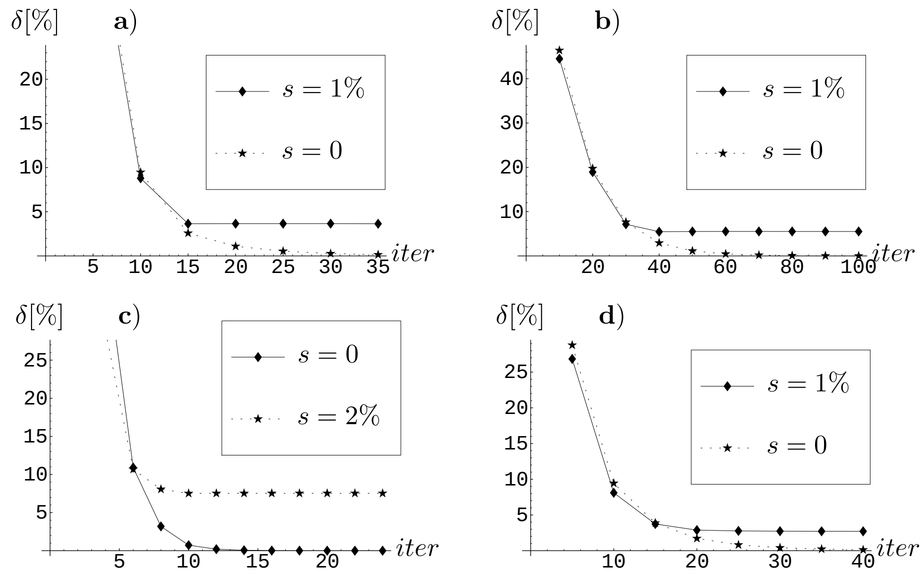

Fig. 8. Relation between the error $\delta[\%]$ and the number of iterations for $n = 20$, $pkt = 18$ in system $(1 \times 1, 1 \times 1)$ with and without the normal noise s obtained in the simulation of algorithms **a**) – *SZB*-3, **b**) – *RB*-3, **c**) – *CHART*-3, **d**) – *CHRB*-3

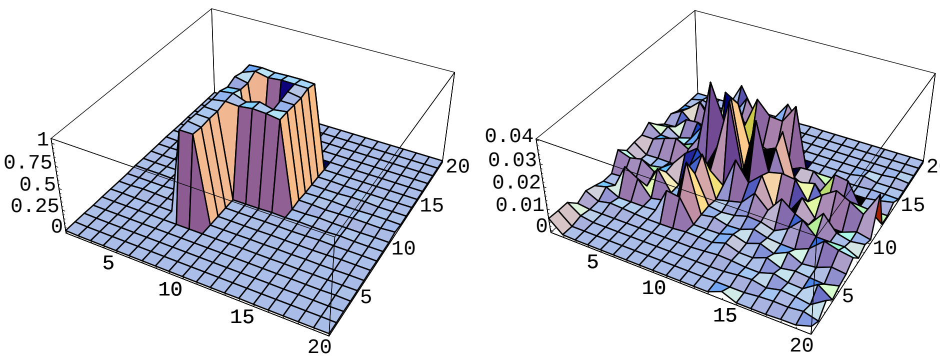

Fig. 9. Reconstruction and its error Δ obtained for $n = 20$, $pkt = 18$ in system $(1 \times 1, 1 \times 1)$ with the normal noise $s = 1\%$ in the simulation of algorithm *ART*

The research has been of course executed for the wide range of elements influencing the results. We have tested various functions describing the density distribution, various densities of discretization and various locations of the non-transparent element. It turned out that in each investigated case (for the non-transparent elements of "reasonable" size) the obtained reconstructions were of the same quality as presented in Figures 9–12. Some selected reconstructions are shown in Figures 13–17.

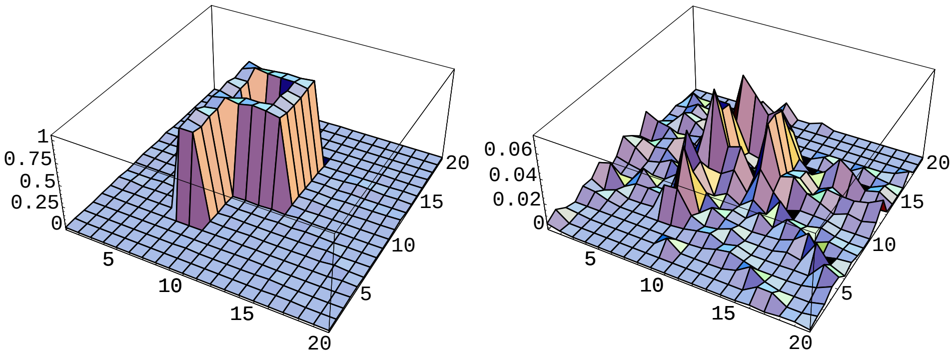

Fig. 10. Reconstruction and its error Δ obtained for $n = 20$, $pkt = 18$ in system $(1 \times 1, 1 \times 1)$ with the normal noise $s = 2\%$ in the simulation of algorithm ART

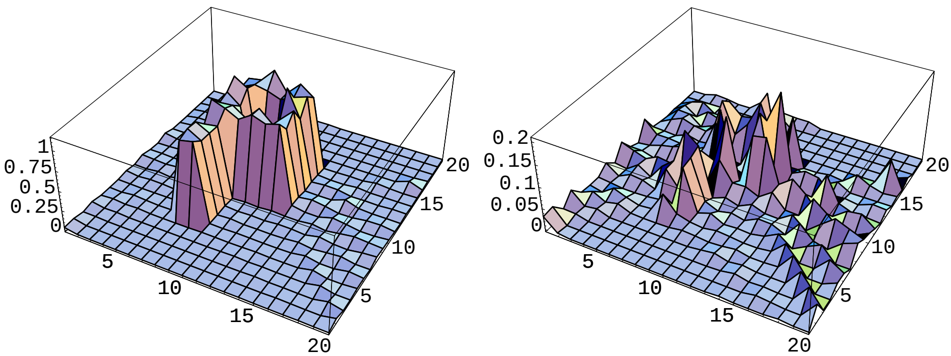

Fig. 11. Reconstruction and its error Δ obtained for $n = 20$, $pkt = 18$ in system $(1 \times 1, 1 \times 1)$ with the normal noise $s = 5\%$ in the simulation of algorithm ART

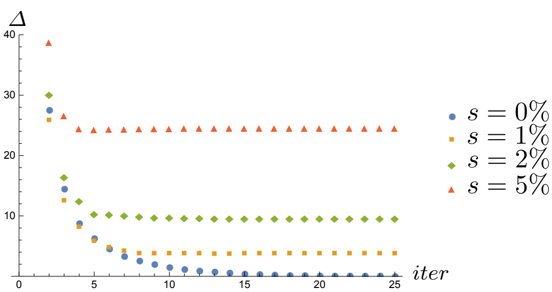

Fig. 12. Relation between the error Δ and the number of iterations for $n = 20$, $pkt = 18$ in system $(1 \times 1, 1 \times 1)$ in the simulation of algorithm ART

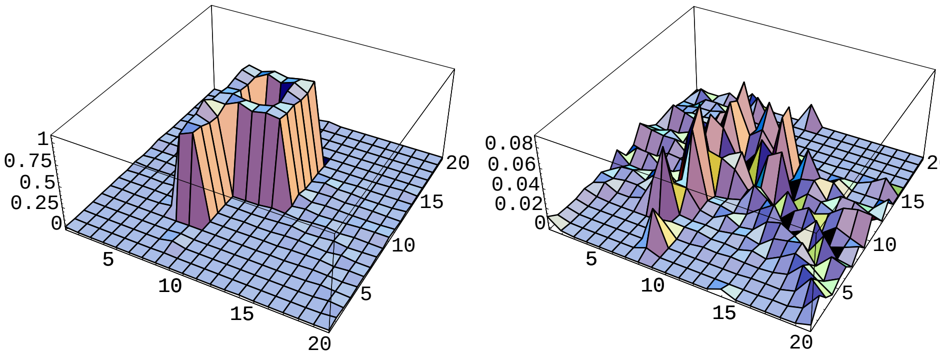

Fig. 13. Reconstruction and its error Δ obtained for $n = 20$, $pkt = 18$ in system $(1 \times 1, 1 \times 1)$ with the normal noise $s = 2\%$ in the simulation of algorithm ART and non-transparent element located at point $(0.5, -0.7)$

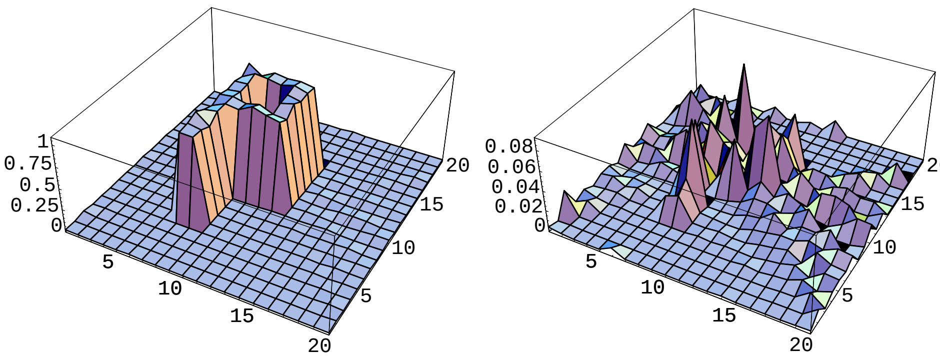

Fig. 14. Reconstruction and its error Δ obtained for $n = 20$, $pkt = 18$ in system $(1 \times 1, 1 \times 1)$ with the normal noise $s = 2\%$ in the simulation of algorithm ART and non-transparent element located at point $(-0.75, -0.75)$

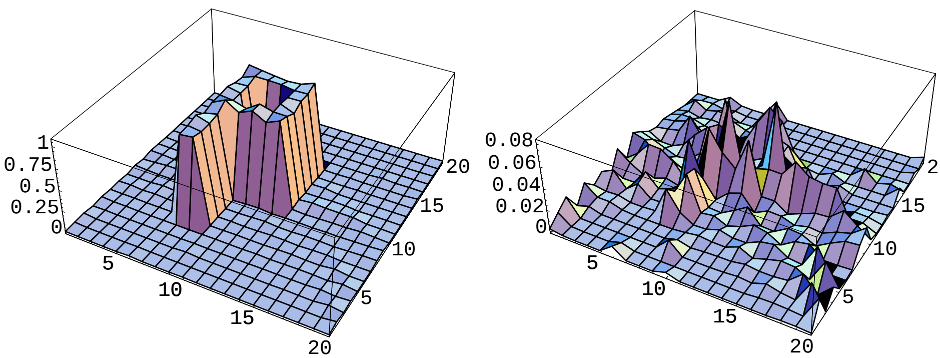

Fig. 15. Reconstruction and its error Δ obtained for $n = 20$, $pkt = 18$ in system $(1 \times 1, 1 \times 1)$ with the normal noise $s = 2\%$ in the simulation of algorithm ART and non-transparent element located at point $(-0.7, 0.6)$

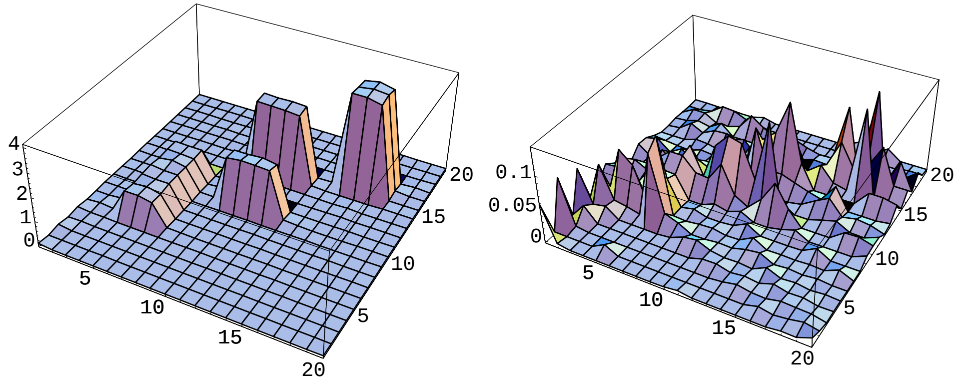

Fig. 16. Reconstruction and its error Δ obtained for $n = 20$, $pkt = 18$ in system $(1 \times 1, 1 \times 1)$ with the normal noise $s = 2\%$ in the simulation of algorithm ART and non-transparent element located at point $(0.5, 0.7)$

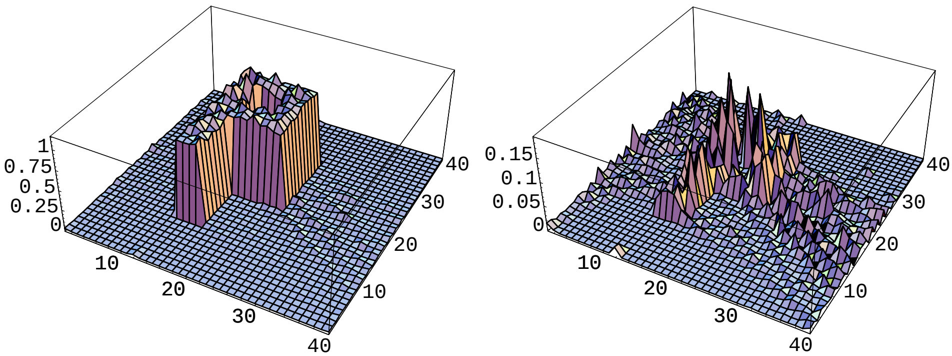

Fig. 17. Reconstruction and its error Δ obtained for $n = 40$, $pkt = 42$, $iter = 50$ in system $(1\times1, 1\times1)$ with the normal noise $s = 2\%$ in the simulation of algorithm ART and non-transparent element located at point $(-0.1, 0.2)$

Similar situation can be observed for system (1×1). Obviously, for obtaining the reconstruction of good quality we need in this case much more iterations (the same observation we have noticed in case of problems with no non-transparent elements).

In Figures 18–20 there are presented the selected reconstructions received by using algorithm ART-3 together with the relations between the error Δ and the number of iterations. And again, we have to notice that the research has been of course executed for the wide range of elements influencing the results, but because of the size of this paper we present only the selected results.

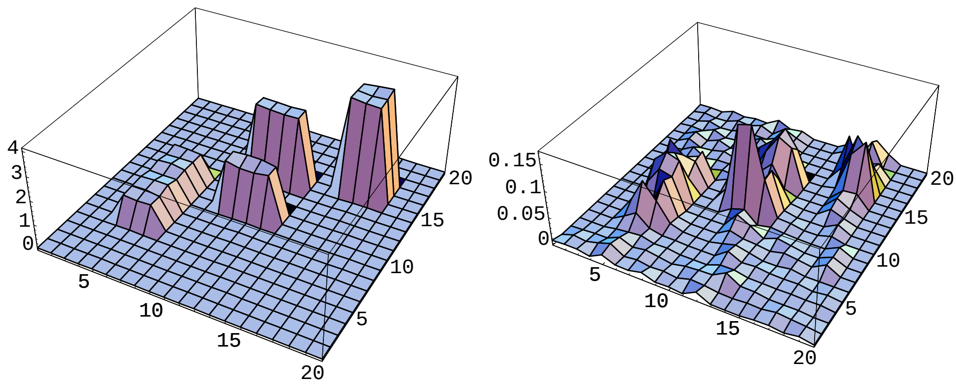

Fig. 18. Reconstruction and its error Δ obtained for $n = 20$, $pkt = 28$ in system (1×1) with the normal noise $s = 1\%$ in the simulation of algorithm ART-3 and non-transparent element located at point $(0.2, -0.6)$

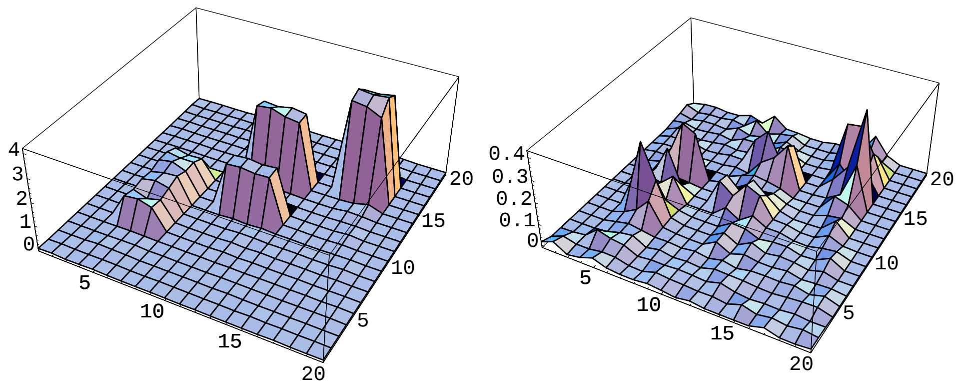

Fig. 19. Reconstruction and its error Δ obtained for $n = 20$, $pkt = 28$ in system (1×1) with the normal noise $s = 2\%$ in the simulation of algorithm ART-3 and non-transparent element located at point $(0.2, -0.6)$

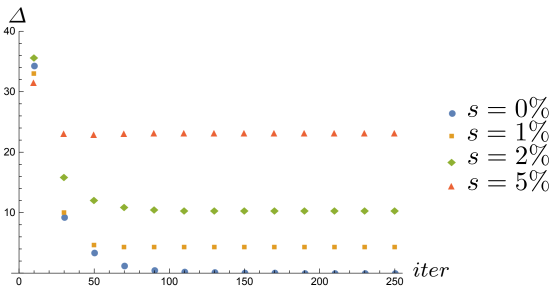

Fig. 20. Relation between the error Δ and the number of iterations for $n = 20$, $pkt = 28$ in system (1×1) in the simulation of algorithm ART-3 and non-transparent element located at point $(0.2, -0.6)$

With the same situation we deal in case of applying the chaotic algorithm. Also for this algorithm we have performed the extensive research showing that the chaotic algorithm is useful in investigating the objects with the non-transparent elements and, similarly like in case of examining the objects without the non-transparent elements – for obtaining the reconstruction of quality comparable to the quality received with the aid of previously discussed algorithms, less iterations is required. Figures 21–23 display the selected reconstructions received by using algorithm *CHART*-3 and the relations between the error Δ and the number of iterations.

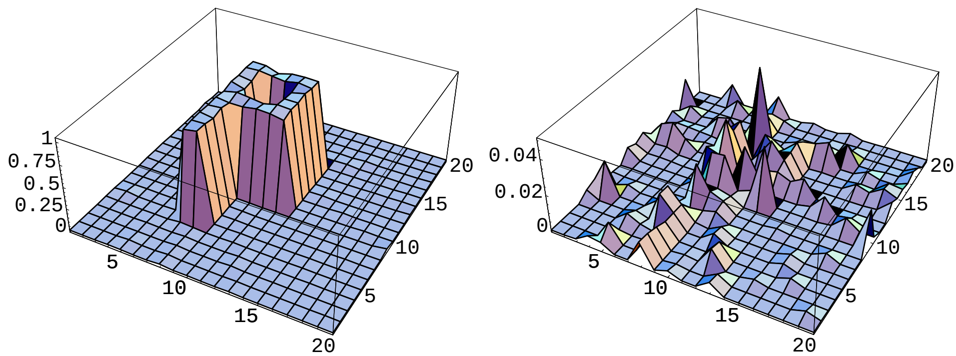

Fig. 21. Reconstruction and its error Δ obtained for $n = 20$, $pkt = 28$ in system $(1\times 1, 1\times 1)$ with the normal noise $s = 1\%$ in the simulation of algorithm *CHART*-3 and non-transparent element located at point $(0.2, -0.6)$

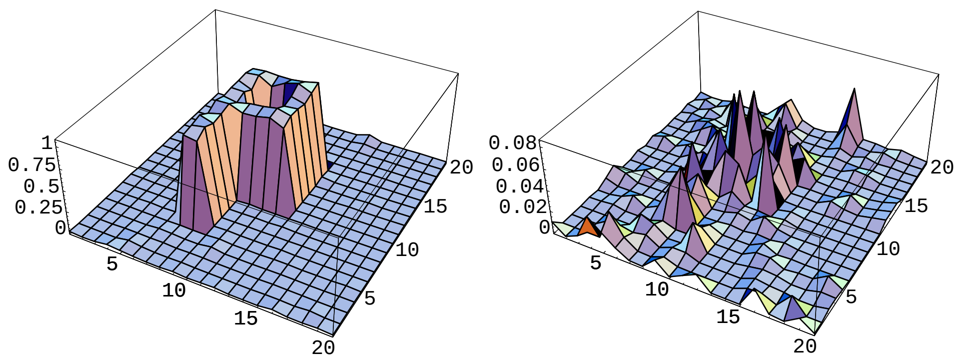

Fig. 22. Reconstruction and its error Δ obtained for $n = 20$, $pkt = 28$ in system $(1\times 1, 1\times 1)$ with the normal noise $s = 2\%$ in the simulation of algorithm *CHART*-3 and non-transparent element located at point $(0.2, -0.6)$

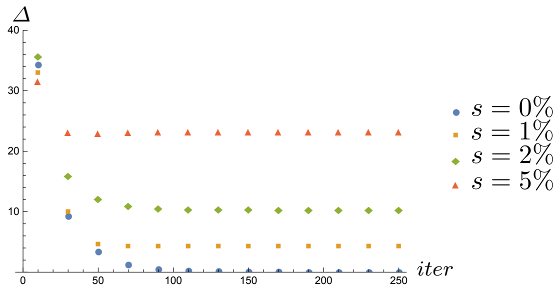

Fig. 23. Relation between the error Δ and the number of iterations for $n = 20$, $pkt = 28$ in system $(1 \times 1, 1 \times 1)$ in the simulation of algorithm *CHART*-3 and non-transparent element located at point $(0.2, -0.6)$

7. Summary

The previous and current research indicate that it is possible to apply the discussed here algebraic algorithms for considering the problems of computer tomography with the incomplete information and also for investigating the objects with the non-transparent elements. The previous research showed the usefulness (convergence and stability) of the described algorithms in solving the problems of incomplete set of data as well as their convergence in solving the problems of incomplete set of data but in case of investigating the objects with non-transparent elements. In the current paper we extended the research for investigation of stability of the discussed algorithms in case of problems with the incomplete information applied for the objects with the non-transparent elements.

This paper is some kind of closure and summary of some stage of research – therefore we did not include here the examination of stability of the block algorithms. In the previous works we have only theoretically tested these algorithms – their parallel run was only simulated sequentially by using just one processor. The next step of our research in this subject will consist in the practical implementation of the parallel algorithms (block and chaotic-block algorithms), thus we plan to investigate the stability of these algorithms by applying the parallel computations.

The next step of our research, planned for the future and independent on the subject matter presented in this paper, is the full 3D reconstruction on the basis of only few investigated layers, in opposite to two approaches considered till now in the classical 3D tomography. We want to realize this reconstruction by applying the Hermite interpolation.

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